ON SIMPLIFIED DESIGN METHODS FOR NONLINEAR DYNAMIC MECHANICAL SYSTEMS

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SYNOPSIS

Dimensional analysis and classical methods are combined to develop a simplified mathematical description of two practical nonlinear dynamic mechanical systems experiencing excitation similar to that due to an earthquake. This method is used to obtain system descriptions in closed algebraic form with appropriate constants. The number of constants provides the designer with the minimum number of complex time-history analyses or tests required to completely characterize the systems. Once these constants are evaluated, parametric and optimization studies can be performed very quickly by hand. Predictions based on application of the method are made. Comparisons of these predictions to results obtained from time-history solutions show the method is a valuable design tool. Application of the method to other situations is briefly discussed.

RESUME

On présente dans cet article une méthode mathématique simple pour étudier deux systèmes mécaniques non linéaires soumis à une excitation dynamique semblable à celles produites par un tremblement de terre. Cette méthode est utilisée pour décrire les systèmes sous forme algébrique, avec des constantes convenables. Lorsque ces constantes ont été déterminées on peut faire rapidement à la main des études paramétriques et des études d'optimisation. On a fait des prédictions en utilisant cette méthode et la comparaison de ces prédictions aux résultats obtenus avec des méthodes plus exactes montre que la méthode proposée dans cet article est un outil très utile pour le dimensionnement. On discute brièvement de l'application de cette méthode à d'autres cas.

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INTRODUCTION

Many practical engineering problems arise during the design phases of a dynamic mechanical system or component with inherent nonlinearities. The designer is often forced to perform a series of complex and expensive time-history analyses or tests to show that the design requirements are met. Then if a major parameter is changed, as often happens, it is necessary to repeat this expensive, time-consuming procedure. Because of the above and the fact that nonlinear dynamic mechanical systems are commonly used for control system applications, simplified design methods applicable to this type of system are particularly appealing.

It is intuitively likely that many mechanical systems are amenable to a simplified description because they possess some or all of the following characteristics:

- the geometric nonlinearity effects are bounded in terms of forces of displacements (or both).

- the time scale to describe the local nonlinear effects is small in comparison to an output parameter time scale so that a time average description of local effects is appropriate.

- the time scale to describe the output parameter is small in comparison with a system characteristic time constant so that higher order terms can be neglected.

- the required output consists of something less than the complete detailed time-history response of the system and can be described with one parameter.

In order to develop the simplified method a dimensional analysis is performed for a system consisting of a lumped mass being inserted into a cylinder that is subjected to a lateral sinusoidial displacement history. This analysis leads to a simple algebraic description of the system in terms of the appropriate physical parameters and system constants. These constants are evaluated by employing results from two of a series of nine time-history analyses of the system. Then, the simplified algebraic description is used to predict the response of the system for the other seven analyses and comparisons are made to the time-history results. These comparisons show that the approach works well for the lumped mass system where the mass is acted on by forces arising from gravitational and impact frictional effects.

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The second system considered consists of a rod inserted into three bushings while the bushings are laterally displaced. This system can be viewed as an extension of the lumped mass system because, in addition to gravitational and impact frictional forces, there exist forces due to an insertion assist spring and fluid drag. Each of the latter forces is initially described by more conventional means. In order to obtain the correct physical parameters and form, a relevant differential equation is solved. The solution is put into series form and the appropriate terms are chosen. These terms are then made dimensionless and inserted into an extension to the simplified algebraic description developed for the lumped mass system. Again, comparison with analytical results shows that predictions made with the extended simplified system description are accurate.

Dimensional Analysis of the Lumped Mass Nonlinear System

The lumped mass in a cylinder problem is shown in Figure 1. The mass falls in the positive y coordinate direction due to the gravitational force. Lateral motion of the cylinder causes impacting which gives rise to a retarding force through friction. The required output is the time necessary for the mass to travel a given distance, the insertion distance, in the y direction. A dimensional matrix for this system is constructed as follows:

	[F]	[L]	[T]
ω	0	0	-1
x_o	0	1	0
G	0	1	0
μ	0	0	0
k	1	-1	0
m	1	-1	2
g	0	1	-2
y	0	1	0
t	0	0	1

Since there are nine quantities and the rank of the matrix is three, there are six dimensionless parameters which will completely describe the system [1][1]. One choice for the dimensionless parameters and their significance is as follows.

$$\text{Input:}\quad \Omega \equiv \frac{\omega}{(k/m)^{\frac{1}{2}}}\,,\ X_o \equiv \frac{x_o\omega^2}{g}\,;\qquad \text{Output:}\quad T \equiv t(k/m)^{\frac{1}{2}}$$

$$\text{System Parameters:}\quad \mu,\ \phi \equiv \frac{GK}{mg}\,,\quad Y \equiv \frac{yk}{mg}\,.$$

[1]Numbers in brackets designate References listed at the end of this paper.

With the above the following mathematical statement can be made

$$\mathcal{F}\ (\Omega,\ X_0, T, \mu,\ \phi,\ Y)\ =\ \text{Constant.}$$

Based upon the similarity between the problem of Figure 1 and that of a freely falling body this relationship is rewritten as

$$Y\ =\ A\ (\mu,\ X_0,\ \Omega,\ \phi) T^2 \quad [0] \tag{1}$$

where $A\ (\mu,\ X_0,\ \Omega,\ \phi)$ is a dimensionless acceleration.

Derivation of the Form of the Dimensionless Acceleration for Gravitational and Frictional Forces

To develop the form of the dimensionless acceleration, $A\ (\mu,\ X_0,\ \Omega,\ \phi)$, that is consistent with equation (1), the expression for a freely falling body is modified to introduce the retarding effect of impact forces due to contact with the cylinder times the coefficient of friction, μ, between the mass and the cylinder.

The free fall expression for a body starting at rest is

$$y\ =\ \frac{1}{2}\, g\ t^2\ [L]\ . \tag{2}$$

If a time average of the impact forces is defined as $\bar{F}$ and introduced into equation (2), then

$$y\ =\ \frac{1}{2}\ (g - \frac{\mu\bar{F}}{m})\ t^2\ [L] \tag{3}$$

where $\frac{\mu\bar{F}}{m}$ is the time average acceleration due to the frictional retardation effect of the impacting. Defining $a = \frac{1}{2}\ (g - \frac{\mu\bar{F}}{m})$, then $a/g = \frac{1}{2}\ (1 - \frac{\mu\bar{F}}{mg})$, and equating this to the dimensionless acceleration gives

$$A\ (\mu,\ X_0,\ \Omega,\ \phi)\ =\ a/g\ =\ \frac{1}{2}\ (1 - \frac{\mu\bar{F}}{mg})$$

or

$$A = \frac{1}{2}\ (1 - \frac{\mu\bar{F}}{mg})\quad [0]\,. \tag{4}$$

To complete the dimensionless acceleration description an expression for the time average force, $\bar{F}$, must be developed using the system parameters. The expression can be obtained with the impulse-momentum relationship $\bar{F}\,\Delta t = m\,\Delta v$.

The total time increment, Δt, for the mass to traverse in the x-direction is proportional to the time increment, Δt_1, to traverse the gap, G, and the time increment, Δt_2, which is the fundamental period of the mass acting on the wall stiffness, k, i.e.,

$$\Delta t \propto \Delta t_1 + \Psi_1\,\Delta t_2 \quad [T] \tag{4a}$$

where Ψ_1 is a weighting factor. The total time period, Δt, must be sufficiently large to describe the time average response due to the impact forces.

From fundamental equations of motion

$$\Delta t_1 = \frac{G}{\omega x_0} \quad [T] \tag{4b}$$

and

$$\Delta t_2 = \left(\frac{m}{k}\right)^{1/2} \quad [T] \quad . \tag{4c}$$

Substituting equations (4b) and (4c) into (4a) gives

$$\Delta t \propto \frac{G}{\omega x_0} + \Psi_1 \left(\frac{m}{k}\right)^{1/2} \quad [T] \quad . \tag{4d}$$

The cumulative change in velocity, Δv, in the x-direction during the time period Δt is described in terms of the frequency, ω, and amplitude, x_0, of the input excitation by the following relationship

$$\Delta v \propto \omega x_0 \quad [L/T] \quad . \tag{4e}$$

Substituting equations (4d) and (4e) into the impulse-momentum relationship and solving for $\bar{F}$ gives

$$\bar{F} = \frac{\Psi_2\, m\, \omega x_0}{G/\omega x_0 + \Psi_1 (m/k)^{1/2}} \quad [F] \tag{5}$$

where Ψ_2 serves to insure that the time period, Δt, is sufficiently large to describe the time average response, as required.

Since the magnitude of $\overline{F}$ is dependent on the time period over which the average is taken until that period becomes large enough so that any further increase will not change the magnitude of $\overline{F}$, and since that minimum time period can be related to the input frequency, it is reasonable to use the following expression for Ψ_2

$$\Psi_2 \equiv C_2\,\omega \quad [0] \quad = \text{"time constant" times the input frequency}$$

and, for the lumped-mass problem,

$$\Psi_1 \equiv C_1 \quad [0] \quad = \text{"system constant".}$$

Substituting for Ψ_1 and Ψ_2 in equation (5) and substituting that result into equation (4) and rearranging gives the following general form for the dimensionless acceleration,

$$A = \frac{1}{2}\left(1 - \frac{C_2\,\omega\,\mu\,(\omega\,x_o)^2}{g\left[G+C_1(m/k)^{1/2}\,(\omega x_o)\right]}\right) \quad [0] \; . \tag{6}$$

Substitution of equation (6) into equation (1) will result in a closed form expression for the nonlinear lumped mass problem in terms of two constants, C_1 and C_2.

Note that if the gap, G, in equation (6) becomes zero, the second term takes the form of an inertial term, i.e., is proportional to $\omega^2 x_o$.

Application to a Lumped Mass System

To demonstrate that equation (6) when coupled with equation (1) provides an adequate description of the nonlinear response of the system of Figure 1, the following approach was employed:

(a) A parametric study of the nonlinear system of Figure 1 was performed for nine different cases using the ANSYS [2] computer code.

(b) The results of two of the nine cases were used to evaluate the "system" constant and the "time" constant in equation (6) to obtain a specific expression for the dimensionless acceleration.

(c) The dimensionless accelerations for the seven remaining cases were calculated using the specific expression obtained in (b) and these were compared to the ANSYS computer results obtained in (a).

Lumped Mass System Parametric Study

The ANSYS general purpose finite element computer code was used to obtain solutions for the system of Figure 1. Table 1 gives a listing of the system parameters which remained constant throughout the study and Table 2 gives a listing of the cases analyzed and the numerical values for the varied parameters for each case. With the addition of the independent variable t (s) and the dependent variable y (in) all quantities of Figure 1 are included in the parametric study. The following correspondence exists between the varied parameters of Table 2 and the dimensionless parameters in equation (1):

Variable (Table 2)	Corresponding Dimensionless Parameter
x_0	$X_0 = \dfrac{x_0\,\omega^2}{g}$
ω	$\Omega = \omega/(k/m)^{1/2}$
μ	μ
G	$\phi = Gk/mg$

Sample results from the parametric study are presented in Figures 2 through 4 for cases 1, 2 and 5, respectively, of Table 2.

Evaluation of the "System" and "Time" Constants

To simplify the numerical calculations two modifications are introduced into equation (6). They are

$$D \equiv C_2/g \quad [T^3\,L^{-1}]$$

and

$$E \equiv C_1(m/k)^{1/2} \quad [T]$$

By introducing these identities equation (6) becomes

$$A = \frac{1}{2}\left(1 - \frac{D\mu\omega^3 x_0^{\;2}}{G + E\omega x_0}\right) \quad [0] \; . \tag{7}$$

This can be rewritten as

$$D\ (\mu\omega^3 x_0^{\ 2}) \ + \ E\ (2A-1)(\omega x_0) \ + \ (2A-1)G \ = \ 0 \quad [L] \quad . \tag{8}$$

The constants D and E can be determined by writing equation (8) using values of A determined from any two of the nine cases of Table 2. For purposes of clarity the following convention is adopted throughout the remainder of this paper;

A is defined to be a dimensionless acceleration calculated by means of the derived empirical equations being developed in this paper.

CDA is defined as a Calculated Dimensionless Acceleration and is obtained either from analytically determined insertion times, t, by means of the relationship CDA = Insertion distance/386$(t)^2$, or from the average slope of the velocity vs. time plots, $\overline{\frac{dv}{dt}}$, by means of the relationship CDA = $\frac{1}{2g}\overline{\frac{dv}{dt}}$. It is easily demonstrated that for the cases of Figures 2 and 4 both methods lead to the same results. In general, these values come from analyses or experiments.

Arbitrarily choosing cases 1 and 5 of Table 2, the values of CDA are obtained from the average slopes ($\overline{\frac{dv}{dt}}$) of Figures 2 and 4 as

$$CDA_1 \ = \ \frac{a}{g} \ = \ \frac{1}{2g}\left(\overline{\frac{dv}{dt}}\right)_1 = 0.273$$

and

$$CDA_5 \ = \ 0.283.$$

Using these values for A in equation (8) along with the numerical values of the other parameters, and solving simultaneously, leads to

$$\begin{aligned} D &= 0.0000582 \ s^3/in \\ E &= 0.0283 \ s. \end{aligned} \tag{9}$$

Predictions and Comparisons

When the numerical values for the constants (equations (9)) are substituted into equation (7), the specific expression for the dimensionless acceleration becomes

$$A \ = \ \frac{1}{2}\left(1 - \frac{0.0000582\ \mu\ \omega^3\ x_0^{\ 2}}{G \ + \ 0.0283\ \omega\ x_0}\right). \tag{10}$$

Equation (10) was used to predict the dimensionless acceleration, A, for cases 2 through 4 and 6 through 9 as listed in Table 2. These predictions are given in column 6 of Table 3.

Column 2 of Table 3 gives the average slopes estimated from the ANSYS velocity vs. time plots. Dimensionless accelerations based on the values of column 2 are given in column 3.

Table 3 also shows the percent difference between the dimensionless acceleration values based on the ANSYS results (column 3) and those predicted by equation (10) (column 6) in column 7. The maximum percent difference is 7.4.

Application to a System Including Fluid Drag and Insertion Assist Spring Forces

Analyses were performed for a system consisting of a rod inserted through three bushings while the bushings are subjected to an in-phase sinusoidal lateral displacement time-history. In addition to the gravitational and frictional forces an additional insertion force exists due to the insertion assist spring shown in Figure 5 and an additional retarding force exists due to fluid drag effects shown in Figure 6. The method used for these analyses was to model the system with the rod in three different positions relative to the bushings. These models were used to obtain the impact forces at each of the three bushings for each of the three positions. The time-histories of the sum of the impact forces at the bushings for the three positions were then compared. Because of the high degree of similarity in these time-histories a single time-history was synthesized to represent the spatial sum of the impact forces, Fn, throughout the insertion.

The synthesized time-history of impact forces was input into a special purpose computer program which integrated the following equation of motion to obtain the insertion time, t,

$$m\,a = mg + K'\,(\Delta - y) - \mu\,Fn - Cv$$

where

m, g, μ and y are as defined in Figure 1

K' = insertion spring stiffness [F/L]

Δ = insertion spring free length minus compressed length [L]

$a = \ddot{y}$ = acceleration $[L/T^2]$

$v = \dot{y}$ = velocity [L/T]

Fn = spatial sum of the lateral impact forces as a function of time[F

C = fluid drag coefficient [F T/L] .

Table 4 lists the nineteen cases considered. For all nineteen cases the lateral input excitation is sinusoidal with a 3g amplitude and a 25 Hz frequency. The radial gap, G, between the rod and bushings is 0.025 in and the rod weight, mg, is 71.5 lb. Also, the free length minus the compressed length of the insertion spring is 27 in and the total insertion distance, y, is 37 in. The variable system parameters and insertion time results are also given in Table 4.

Since, for the rod in the three bushings problem, only one parameter, μ, in the second term on the right hand side of equation (10) was varied, there is not enough information to write equations (8)to solve for the constants D and E. However, the factor $D\omega^3 x_o^{\ 2}/G + E\,\omega x_o$ is a constant for the data given and only one equation is required to evaluate it.

Defining this constant as D' and solving equation (10) gives

$$D' = D\omega^3 x_o^{\ 2}/G + E\omega x_o = (1-2A)/\mu .$$

Using the values of CDA = 0.401 for A and μ = 0.20 (corresponding to case 3 of Table 4) the numerical value for D' is obtained as 0.990. Substituting this value into equation (10) gives

$$A = \frac{1}{2}\ (1 - 0.99\mu) \quad . \qquad (10a)$$

Equation (10a) was used to obtain the values of A for cases 1 through 8 of Table 4.

Extension to Include the Insertion Spring Force

To extend the proposed method to include the cases in Table 4 with an insertion spring the previously derived empirical equation was modified to include a spring force term.

Applying Newton's second law to the system of Figure 5 the following differential equation of motion is obtained

$$m\ddot{y} + K'y = m\,g + K'\,\Delta .$$

Direct substitution will show that the solution to this equation is

$$y = (\frac{mg}{K'} + \Delta)\ (1-\cos\ (\frac{K'}{m})^{1/2}t) \quad .$$

By replacing the cosine term in the above expression by its series expansion the following form is obtained

$$y = \frac{1}{2}\left(\frac{mg}{K'} + \Delta\right) t^2 \left(\frac{K'}{m} - \frac{1}{12}\left(\frac{K'}{m} t\right)^2 + \frac{1}{360}\left(\frac{K'}{m}\right)^3 t^4 - \ldots\right). \quad (11)$$

If only the first term of the series is considered equation (11) reduces to

$$y = \frac{1}{2}\left(g + \frac{K'\Delta}{m}\right) t^2 \quad . \quad (12)$$

Examination of equation (12) indicates that for K' or Δ (or both) equal to zero it reduces to the simple expression for a freely falling body.

From equation (12) the dimensionless acceleration for the insertion spring assisted problem can be approximated as

$$a/g = \frac{1}{2}\left(1 + \frac{K'\Delta}{mg}\right) \quad (13)$$

where $K'\Delta/mg$ is the insertion spring term. Introducing the insertion spring term with a "fitting" constant, S, into equation (7), the expression for the dimensionless acceleration becomes

$$A = \frac{1}{2}\left(1 + \frac{S\,K'\Delta}{m\,g} - \frac{D\,\mu\,\omega^3\,x_o^2}{G + E\,\omega\,x_o}\right). \quad (14)$$

Application to Cases with a Spring Force

Table 4 gives four cases (9 through 12) with an insertion assist spring and without fluid drag forces. The purpose here is to show that by evaluating the dimensionless constant, S, in equation (14) it is possible to make accurate predictions of the analytical results.

Since the system parameters are the same as those for the previous cases of Table 4, with the exception of the insertion spring, the constant D' in equation (10a) takes the same value as before, i.e.

$$D' = 0.990 \quad .$$

The insertion assist spring parameters used in the analyses of Table 4 were given as

$$K' = 12 \text{ (lb/in)}, \quad \Delta = 27 \text{ (in)} \quad .$$

Thus, the insertion spring term in equation (14) becomes

$$S \frac{K'\Delta}{m\ g} = S \frac{12 \times 27}{71.5} = 4.53\ S \quad .$$

The first spring assisted insertion case from Table 4 is with $\mu = 0.0$. Calculating CDA from the data for this case gives,

$$CDA = \frac{a}{g} = \frac{y}{g\ t^2} = \frac{37}{386\ (0.21)^2} = 2.17$$

Substituting the above value for A in equation (14) with $\mu = 0.0$ gives

$$2.17 = \frac{1}{2}\ (1 + 4.53\ S) \quad ,$$

or

$$S = 0.737 \quad .$$

Thus, when all the constants are substituted into equation (14) the following expression is obtained

$$A = \frac{1}{2} \left(1 + 0.737\ \left(\frac{K'\Delta}{m\ g}\right) - 0.990\mu \right) . \tag{14a}$$

Predictions made with the above expression are listed in Table 4 for cases 9 through 12.

Extension to Include the Fluid Drag Force

The purpose here is to extend the applicability of the previously derived empirical equation by including the effects of fluid drag. The case of fluid forces proportional to velocity is considered because the analytical results of Table 4 were obtained for that case.

Applying Newton's second law to the system of Figure 6 the following differential equation of motion is obtained

$$m\ \ddot{y} + C\ \dot{y} = m\ g \quad .$$

Direct substitution will show that the solution to this equation is

$$y = g \frac{m}{C} \left(\frac{m}{C} \left(e^{\left(\frac{-Ct}{m}\right)} - 1\right) + t \right) .$$

By replacing the exponential with its series expansion, the following expression is obtained

$$y = \frac{1}{2} gt^2 - \frac{1}{6} \frac{gC}{m} t^3 + \frac{1}{24} g \left(\frac{C}{m}\right)^2 t^4 + \ldots\ldots \quad . \tag{15}$$

It is noted that the above expression reduces to that for the free fall case when C = 0.0.

If only the first two terms of equation (15) are considered, then

$$y = \frac{1}{2} gt^2 - \frac{1}{6} \frac{gC}{m} t^3 \quad . \tag{16}$$

Table 4 provides analytical predictions of drop times where the fluid drag force is proportional to velocity (see cases 13 through 16). The drag force at a bushing with no lateral excitation is [3]

$$F = \nu A \frac{d\varepsilon_i}{dt}$$

where

ν = viscosity of fluid $[FT\ L^{-2}]$

A = area over which shear forces exist $[L^2]$

$\frac{d\varepsilon_i}{dt}$ = strain rate of fluid $[T^{-1}]$.

For three bushings the total drag is

$$F_D = \nu \sum_{i=1}^{3} A_i \frac{d\varepsilon_i}{dt}$$

where

$$\frac{d\varepsilon_i}{dt} = \left(\frac{v}{G}\right)_i \text{ and } A_i = \pi D \ell_i$$

and

v = velocity of rod with respect to the fluid
D = diameter of rod
ℓ_i = length of bushing
G_i = nominal radial gap between rod and bushing.

Therefore

$$F_D = \pi \ D \nu v \sum_{i=1}^{3} \frac{\ell_i}{G_i} \quad . \tag{17}$$

Since all factors on the right hand side of equation (17), except velocity, are constant for a given system this equation can be rewritten as

$$F_D = C v. \tag{18}$$

This drag force is identical to that of Figure 6 which leads to a cubic equation in time, i.e., equation (16). Also, it is the case considered in Table 4 as mentioned earlier. Therefore, to describe this case it is reasonable to proceed as follows. Assume

$$v (t) \equiv \xi (t) \overline{v} \equiv \xi (t) (\frac{y}{t}) \tag{19}$$

where

$\xi (t)$ = an arbitrary dimensionless function of time and

$\overline{v} = \frac{y}{t}$ = average velocity for the insertion.

If the form of ξ is chosen as

$$\xi (t) = \alpha t - \beta t^2 \tag{20}$$

then the following consequences hold:

a) The fluid drag force would be comprised of two terms, one constant and one decreasing linearly with time.

b) A realistic description of the velocity vs. time plots obtained from the cases in Table 4 is possible.

c) It is necessary to solve a cubic equation in time for the insertion time. This is consistent with equation (16).

Based on the above, the dimensionless acceleration, equation (14), is expanded to include the fluid drag effects as follows

$$A = \frac{1}{2} \left(1 + \frac{SK'\Delta}{m g} - \frac{D \mu \omega^3 x_o^2}{G + E\omega x_o}\right) \left(1 - f_1 C + f_2 Ct \right) \tag{21}$$

where

f_1 = fitting constant for the "constant" fluid force term
f_2 = fitting constant for the time dependent fluid force term.

Application to Cases with Spring Force and Fluid Forces

Table 4 provides six cases where the coefficient of friction was taken equal to zero and the drag coefficient (C in equation (18)) was taken as 0.1, 0.5, 1.0, 2.0, 2.5 and 3.0 respectively (see cases 13 through 18).

To evaluate the fitting constants, f_1 and f_2, of equation (21) two sets of data must be used. Choosing cases 14 and 17 from Table 4 for this purpose gives the following simultaneous equations in terms of the fitting constants

$$
\begin{aligned}
1.68 &= \frac{1}{2}\left(1 + 3.35\right)\left(1 - 0.5f_1 + 0.5\ (0.239)\ f_2\right) \\
0.253 &= \frac{1}{2}\left(1 + 3.35\right)\left(1 - 2.5\ f_1 + 2.5\ (0.616)\ f_2\right) .
\end{aligned}
\tag{22}
$$

The solution to equations (22) is

f_1 = 0.518
f_2 = 0.267.

Using these numerical values equation (21) becomes

$$
A = \frac{1}{2}\left(1 + 3.35\right)\left(1 - 0.518\ C + 0.267\ Ct\right) . \tag{23}
$$

Using equation (23) the predicted values of the dimensionless acceleration can be calculated and these are given in Table 4 for cases 13 through 18.

Application to a General Case

Case 19 given in Table 4 is general in that all the forces described in equation (21) are acting. These include

- mechanical drag force due to impacting and friction
- fluid drag force
- gravitational force
- insertion spring force.

Table 4 provides the parameters as used for this general case and the insertion time as calculated by the special purpose computer program. From these, the values for the quantities in equation (21) are calculated as follows

$$\frac{SK'\Delta}{m\ g} = 3.35$$

$$\frac{D\ \mu\ \omega^3 x_o^{\ 2}}{G + E\omega x_o} = D'\ \mu = 0.0990$$

$$f_1C = 0.518\ (1.0) = 0.518$$

$$f_2Ct = 0.267\ (1.0)\ (0.287) = 0.0766 \quad .$$

Substituting the above values into equation (21) gives

$$A = \frac{1}{2}\left(1 + 3.35 - 0.0990\right)\left(1 - 0.518 + 0.0766\right) = 1.19 \quad .$$

This predicted value for the dimensionless acceleration can be compared to the value obtained directly from the data as

$$CDA = \frac{y}{gt^2} = \frac{37}{386} \times \frac{1}{(.287)^2} = 1.16 \quad .$$

The percent difference in these values is less than 2.6.

Accuracy in the Predictions

Tables 3 and 4 give numerical values for both calculated dimensionless accelerations, CDA (based on analytical results), and dimensionless accelerations, A, obtained through the application of the empirical equations developed herein. Comparisons indicate that, for the problems considered, the simplified empirical equations give accurate predictions of the dimensionless accelerations. Since insertion times are inversely proportional to the square root of the dimensionless accelerations, the error in the predicted insertion times is approximately one half the error in A. The largest percent difference in insertion time predictions for the cases considered in this work is 3.7.

Applicability and Extensions of the Methods

Table 2 implicitly gives the range for each of the varied parameters in the parametric study of the lumped mass system. Because of the high degree of accuracy in the predicted dimensionless accelerations it is concluded that the simplified methods used in making the predictions are applicable over the variable ranges addressed. However, there may be finite limits on these parameter ranges which are problem dependent. Therefore, if the methods are applied to other systems, it will be necessary to confirm that the resulting description is accurate over the chosen range of the variables. In practice this verification can be obtained through a judicious choice of the "analyses" or "tests" performed to fit the constants in the system description (derived empirical equation) and, in the worst case, may introduce the requirement that more than the minimum number of "tests" be performed or that more than one equation, each applicable over a certain range of variables, be developed.

The high degree of accuracy in the predicted dimensionless accelerations of Table 4, for the rod in three bushings system, indicates that the linearity assumptions and truncation of terms in equations (11) and (15) are acceptable for this particular system. Again, the acceptability of this method is problem dependent. If the solution time is small in comparison to a system characteristic time then this method will prove to be successful. This, too, must be verified on a case by case basis.

Because of the success of the methods developed in this paper it is appropriate to consider other extensions. Inclusion of terms to describe the effects of other forces, such as buoyancy, is straightforward. More recent work suggests that equation (21) is applicable to cases where fluid forces are proportional to velocity squared and where both velocity and velocity squared fluid forces coexist. Also, it has been shown using test data that by modifying the exponent on x_o in the numerator of the second term in equation (10) rod flexibility effects can be accurately described. Finally, a word should be said about the possibility of using these methods for the case of earthquake excitation. If a "design" earthquake is chosen and represented in the manner discussed in [4] , then each sinusoidial term in the "design" earthquake would lead to one term in equation (10). If the earthquake is filtered through structures then it seems reasonable that a small number of terms would be required. For either case an equation similar to (10a) may be appropriate.

CONCLUSIONS

The work presented in this paper shows that it is possible to use a derived empirical closed form expression to accurately describe certain nonlinear systems. For the problems considered the methods lead to essential forms which expose the minimum number of tests or nonlinear analyses required to complete that description. The methods are general and could be applied to other nonlinear systems.

By using the resulting simplified description of the nonlinear systems the effects of varying any of the system parameters can be easily evaluated by the designer and accurate, hand calculated predictions can be made.

ACKNOWLEDGEMENTS

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REFERENCES

1. Mark's, Standard Handbook for Mechanical Engineers, seventh edition, pp 3-76, McGraw-Hill, New York, 1967.

2. DeSalvo, G. J. and, Swanson, J. A., ANSYS Engineering Analysis System User's Manual, Swanson Analysis Systems Inc., Elizabeth, Pa., 1975.

3. Streeter, V. L., Fluid Mechanics, fourth edition, pp 5, McGraw-Hill, New York, 1966.

4. Romstad, K. M., Bruce, J. and Hutchinson, J. R., "Site Dependent Earthquake Motions, "Journal of the Geotechnical Engineering Division", Proceedings of the American Society of Civil Engineers, Vol. 104, No. GT 11, November, 1978.

TABLE 1: PARAMETRIC STUDY CONSTANTS

Unchanging System Parameter	Value	Units
m	0.185	$lb\text{-}s^2\text{-}in^{-1}$
k	1×10^6	$lb\text{-}in^{-1}$
g	386	$in\text{-}s^{-2}$

TABLE 2: PARAMETRIC STUDY CASES AND VARIED PARAMETERS

Case	x_o (in)	No. of g's	$\omega\ (\frac{rad}{s})$	μ (0)	G (in)
1	0.0443	2	132	0.3	0.008
2	0.0443	2	132	0.1	0.008
3	0.0443	2	132	0.5	0.008
4	0.0222	1	132	0.3	0.008
5	0.0443	2	132	0.3	0.016
6	0.0869	2	94	0.3	0.008
7	0.0313	2	157	0.3	0.008
8	0.0111	0.5	132	0.3	0.008
9	0.0222	1	132	0.01	0.008

TABLE 3: COMPARISON OF DIMENSIONLESS ACCELERATIONS FROM ANSYS ANALYSES (COLUMN 3) AND PREDICTIONS (COLUMN 6) FOR THE LUMPED MASS PARAMETRIC STUDY

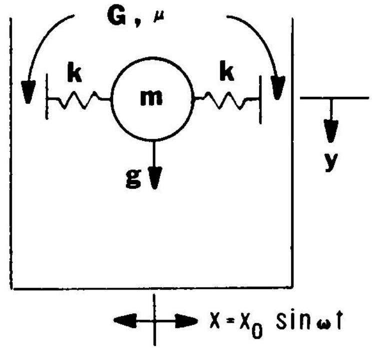
1	2	3	4	5	6	7
Case (See Table 2)	$\frac{\overline{dv}}{dt}$ (plots)	CDA = $\frac{1}{2g}\frac{\overline{dv}}{dt}$	$D\mu\omega^3 x_o^2$	$G+E\omega x_o$	Dimensionless Acceleration, A (Equation (10))	% Difference
1*	211.	0.273	0.0789	0.174	0.273	0.0
2	330.	0.427	0.0263	0.174	0.424	0.7
3	101.	0.130.	0.131	0.174	0.122	6.2
4	300.	0.389	0.0197	0.0908	0.392	-0.8
5*	219.	0.283	0.0197	0.0454	0.283	0.0
6	219.	0.283	0.110	0.240	0.270	4.6
7	214.	0.278	0.0663	0.147	0.275	1.1
8	167.	0.432	0.00492	0.0494	0 450	-4.2
9	179.	0.462	0.000657	0.0909	0.496	-7.4

*These cases were used to evaluate the "time" and "system" constants.

TABLE 4: ROD IN THREE BUSHINGS PROBLEM CASES, VARIED PARAMETERS AND RESULTS

Case	μ	K'(lb/in)	C(lb-s/in)	Insertion Time, (s)	CDA $(37/386(t)^2)$	A	(equation number)
1	0.00	0.0	0.0	0.438	0.500	0.500	(10a)
2	0.10	0.0	0.0	0.461	0.451	0.451	(10a)
3*	0.20	0.0	0.0	0.489	0.401	0.401	(10a)
4	0.30	0.0	0.0	0.522	0.352	0.352	(10a)
5	0.50	0.0	0.0	0.616	0.253	0.253	(10a)
6	0.70	0.0	0.0	0.793	0.152	0.154	(10a)
7	0.75	0.0	0.0	0.867	0.128	0.129	(10a)
8	0.80	0.0	0.0	0.967	0.103	0.104	(10a)
9*	0.00	12.0	0.0	0.210	2.17	2.17	(14a)
10	0.10	12.0	0.0	0.212	2.13	2.12	(14a)
11	0.50	12.0	0.0	0.223	1.93	1.93	(14a)
12	1.00	12.0	0.0	0.241	1.65	1.68	(14a)
13	0.00	12.0	0.1	0.215	2.07	2.07	(23)
14*	0.00	12.0	0.5	0.239	1.68	1.68	(23)
15	0.00	12.0	1.0	0.281	1.21	1.21	(23)
16	0.00	12.0	2.0	0.460	0.453	0.455	(23)
17*	0.00	12.0	2.5	0.616	0.253	0.253	(23)
18	0.00	12.0	3.0	0.780	0.158	0.154	(23)
19	0.10	12.0	1.0	0.287	1.16	1.19	(21)

*These cases were used to numerically evaluate constants (see text).



x_o = max. input amplitude [L]*

ω = input displacement frequency $\left[\frac{1}{T}\right]$

k = impact spring constant $\left[\frac{F}{L}\right]$

m = mass of falling body $\left[\frac{FT^2}{L}\right]$

g = gravitational constant $\left[\frac{L}{T^2}\right]$

μ = coefficient of friction [0]

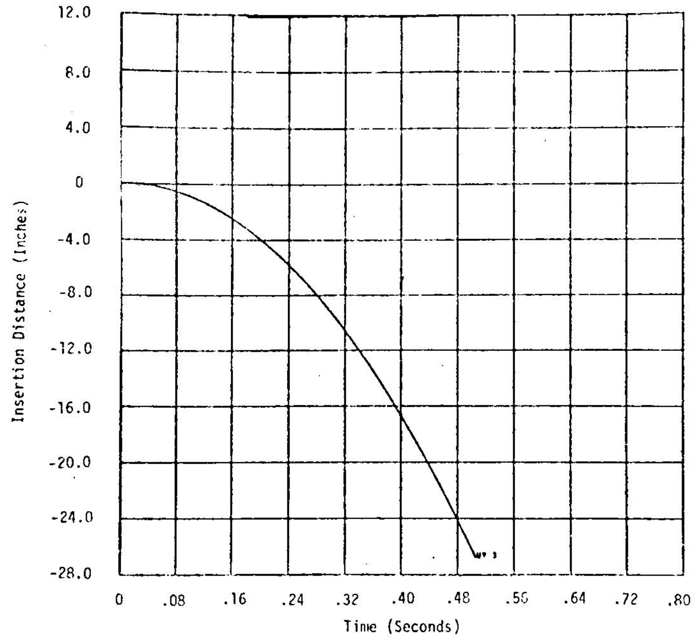
G = radial gap [L]

y = insertion distance [L]

t = time [T]

*Symbols in brackets designate dimensions and [0] indicates a dimensionless quantity.

Figure 1. Nonlinear Lumped Mass System



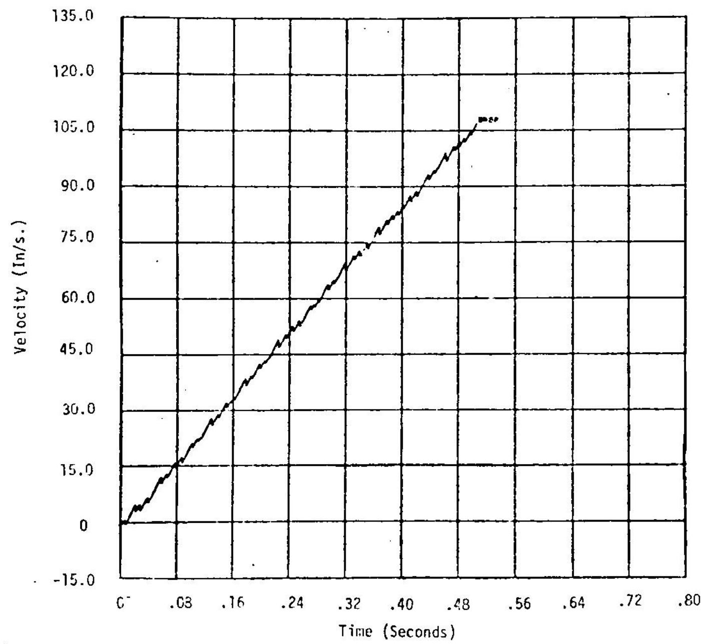


Figure 2

Case 1, Insertion Distance & Velocity vs. Time

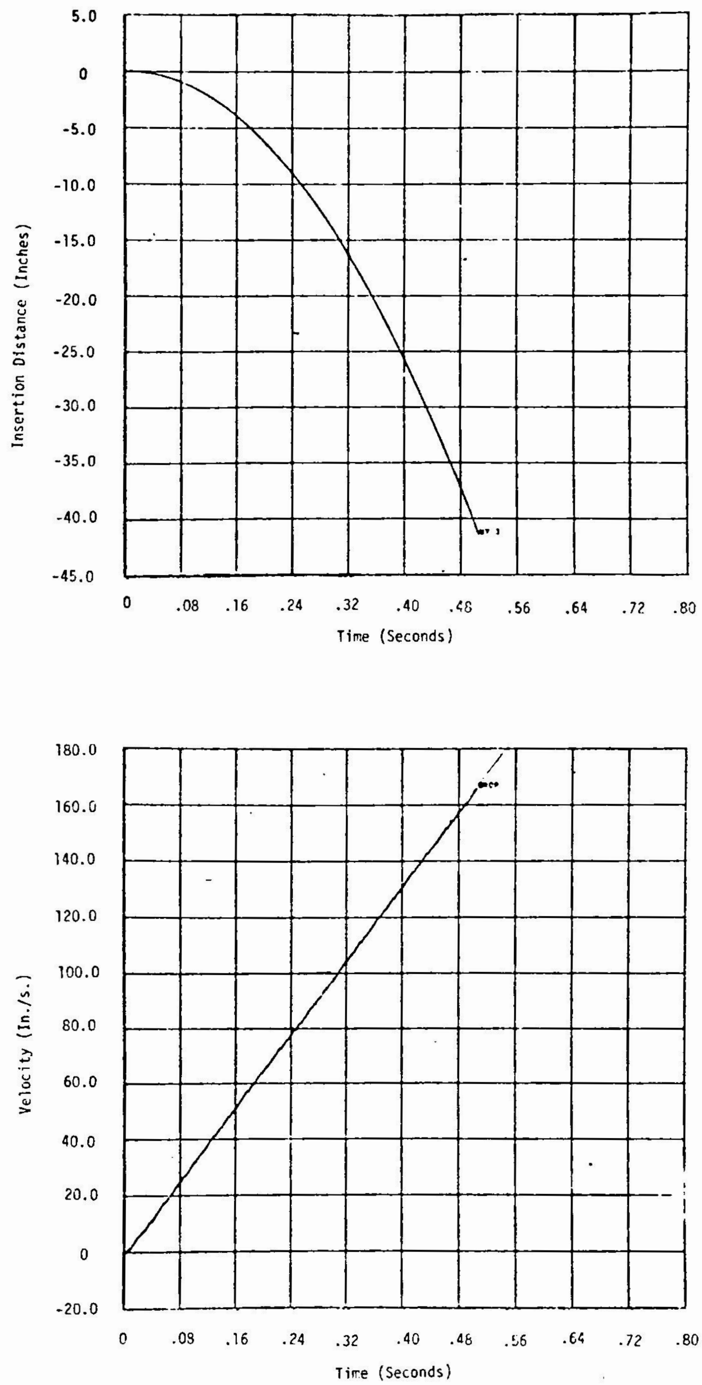
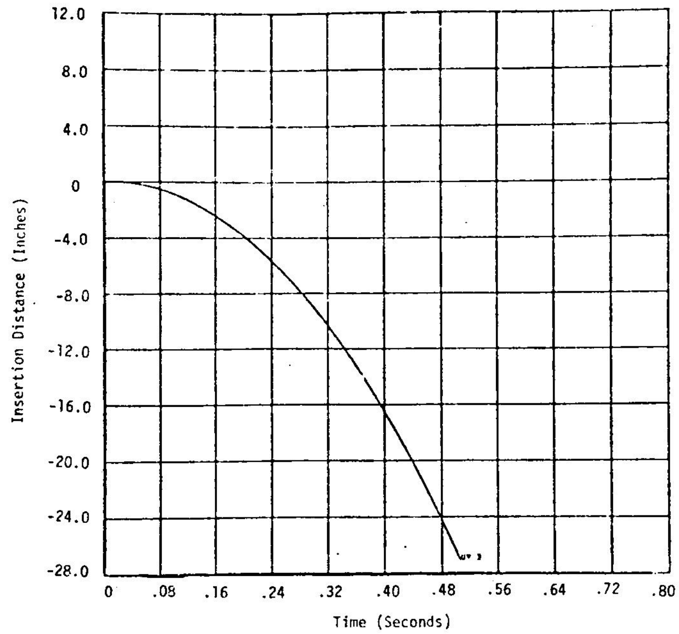


Figure 3

Case 2, Insertion Distance & Velocity vs. Time



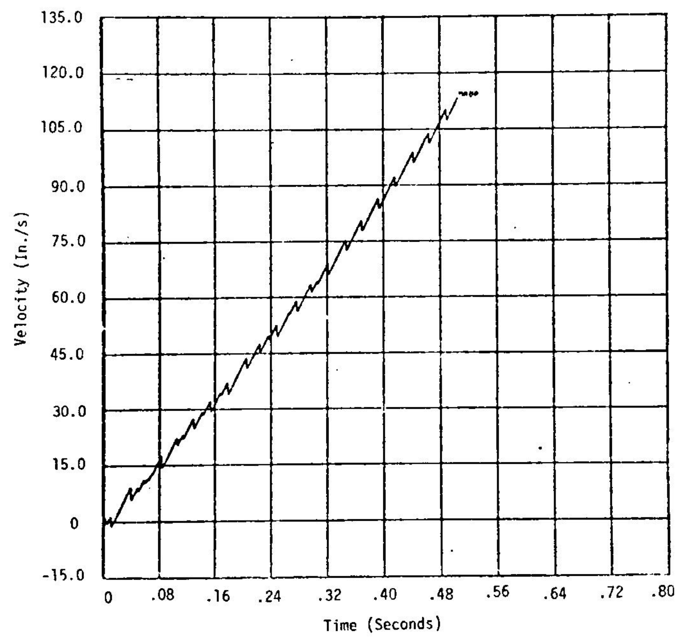
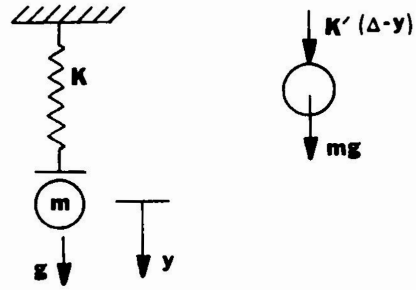


Figure 4

Case 5, Insertion Distance & Velocity vs. Time



Δ = Spring free length - Compressed length   K′ = Spring constant

Figure 5. Falling Mass with Insertion Assist Spring

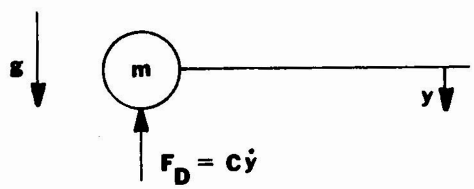


Figure 6. Falling Mass with Fluid Drag Force (F_D)